On the Reduction of Positive Quadratic Form with three Indeterminate Integers

Gustav Lejeune Dirichlet

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To God

Preface

Johann Peter Gustav Lejeune Dirichlet was born on 13^{th} February 1805 , in Düren of French Empire. He died on 5^{th} May 1859 , in Göttingen, Hanover. The young from Richelet or *Le jeune de Richelet*, for the town in Belgium where his family came from, he is not from France as many had claimed.

In his youth Dirichlet showed interests in history and mathematics. He treasured his copy of Gauss's Disquisitiones arithmeticae as others might a bible. When Gauss died in 1855, he was offered his chair at Göttingen. With him the golden age of mathematics in Berlin began. His proofs are characterised by surprisingly simple initial observations followed by extremely sharp analysis of the problem.

A part of Dirichlet's works has become historically important in the study of Voronoi Tessellation. A tessellation is an aggregate of cells that cover the space without overlapping. A Voronoi polygon is also known as a Dirichlet polygon, a Wigner-Seitz polygon, a Theissen polygon, a Blum's transform, an S polygon, a cell model, a plant polygon, Wirkkungsbereich, etc.

Definition 1. *Let Φ be a distribution of a countable set of nuclei $\{x_i\}$ in R^d, and let $x_1, x_2, x_3, \ldots$ be the coordinates of the nuclei. Then, the Voronoi region is*

$$\Pi_i = \{x | d(x, x_i) < d(x, x_j) \forall j \neq i\}$$

where $d(x, y)$ is the Euclidean distance between x and y.

Voronoi tessellation is the solution of a proximity problem, namely the division of the space into n partitions around n particles, such that all points within the i^{th} partition is closest to the i^{th} particle than any other particle. There are a host of proximity problems which, in the end, are related to one another and to the

Voronoi problem. Some example of these are the problems concerning the nearest neighbour, the closest pair and the Euclidean minimum spanning tree. The minimum spanning tree always contains the shortest edge of the graph.

Given a convex hull containing n points, one can join all the points together by straight line segments such that the whole region inside the convex hull is tessellated by the triangles formed by them. This problem is related to the nearest neighbour problem, since among all straight lines connecting to each point there is one which joins it to its nearest neighbour. Moreover, the problem is related to a problem of spatial proximity the solution of which is the Voronoi tessellation. The solutions of these two problems are dual to each other. The triangular tessellation is called the Delaunay triangulation.

Descartes was the first person to draw a picture of a Voronoi tessellation (Descartes, 1644)[†] In his essay, he imagine vorticities surrounding heavenly bodies. The path of an object through space, he says, passes along edges and vertices of what is now known as the Voronoi tessellation. The idea he introduced was original but the discourse philosophical, which is perhaps why his name was never associated with the tessellation which could easily have borne his name instead of that of Voronoi.

Even though we regard Philosophy very highly as the mother of all sciences, as is probably the reason why we call our highest formal education 'Doctor of Philosophy' or Ph.D., but from our experience we could see that philosophy in our dictum generally means only one thing, that is mathematics. Therefore the tessellation is named after Voronoi because he was the first person to have written a *substantial* amount of mathematics on it.

In short, Descartes has provided the idea and philos-

[†] Renati Des-cartes, *Principia philosophiæ*, 1644. Ludovicum Elzevirium (*cf* René Descartes, *Œuvres philosophiques*, Volume III, Garnier Frères, Ferdinard Alquié, Ed., 1973.

ophy, Dirichlet the geometrical description and Voronoi the mathematics. Voronoi could easily have claimed having written the most amount of mathematics on the tessellation which now bears his name, than any other person to date. Ironically it was Descartes' philosophy that all knowledges must be based on mathematics, so he should not have minded.

The work contained in the present volume appeared as G. Lejeune Dirichlet, 'Über die Reduction der positiven quadratischen Formen mit drei unbestimmten ganzen Zahlen', *Crelle*, aka *Journal für die reine und angewandte Mathematik*, Volume 40, Pages 209–227, 1850.

The present translation first appeared in *Voronoi Translated, Introduction to Voronoi Tessellation and Essays by G. L. Dirichlet and G. F. Voronoi* (Bangkok, 2001) by Kit Tiyapan, then in *Percolation within Percolation and Voronoi Tessellation* (Bangkok, 2003 and revised edition 2005). It also appears in Tiyapan's Ph.D. thesis (University of Manchester, 2004).

Kit Tyabandha, Ph.D.
England and Thailand, 2007

On the reduction of positive quadratic form with three indeterminate integers

([Lecture in
physical- mathematical class meeting
of the academy,
on $31^{st} July$, 1848 †])

[by G. L. Dirichlet]

[translated by K N Tiyapan]

It is well known that Lagrange had pointed out for the first time that every binary quadratic form reduces, ie. can transform into another equivalent one the coefficients of which satisfy certain inequality conditions, and at the same time had proven that in every class of positive forms there always exists only one such form, so that in this case the various values of a given determinant corresponding to reduced forms can serve as the representatives of the different classes. Later on after in the "Disquisitiones arithmeticae" the ternary form were looked at from a general point of view did it become necessary for the further development of this theory to extend the study for the positive binary forms by Lagrange to the ternary ones, ie. to find out such inequality conditions between the coefficients that would satisfy one and only form in all classes.

This expansion linked with great difficulties is achieved by *Seeber* in a work specifically devoted to the positive ternary forms, the principal contents of which settles it and which *Gauss* characterises in a most interesting announcement ‡ as follows:

† From this treatise an excerpt has already been given in the monthly report, in which the principle of this new treatment of the positive ternary form, the examinations of successive minima, is outlined and the proof of the first one of the two results from Seeber after this principle completely carried out (page 21 of the second Volume of this issue of G. Lejeune Dirichlet's works.)

‡ Crelle's Journal, V. 20, p. 312

We must do full justice to the spirit of the thoroughness by which these facing ‡ us have gone through, and when we for all that have to feel sorry that a great and perhaps much discouragingly complicated nature is attached to it, here the solution of the problem takes 41 pages and the proof 91 pages, thus we will see this by no means as a respected criticism. If a difficult problem or theorem to solve or to prove exists, then the first seeming idea is always to be recognised as a step that a solution or a proof has been found after all, and the question whether this were not of an easier and simpler way would be possible as long as in so doing such a futile question is not considered as of practicability. Therefore we look upon it as untimely to dwell on this question.

The great complication of Seeberian method has for a longtime stimulated me to set up the theory of reduced ternary forms by a simpler method. As I now allow myself to communicate to the class the result of my effort directed towards this, I think in the interest of briefness and, if I could say so, of the lucidity of the presentation, to have to abide by the geometrical form, in which I have conducted the investigation to which I have laid down as basis the noteworthy relations which occur among the quadratics with two or three elements and with known spatial forms. I begin with the explanation of the outline already given by Gauss in the mentioned announcement on these relations.

§1

The ternary form:

$$ax^2 + by^2 + cz^2 + 2a'yz + 2b'xz + 2c'xy = \varphi, \qquad (1)$$

in which we regard x, y, z as first, second, third element, called positive when φ does not become negative for real

‡ The solution of the problem namely to find a reduced form for every class and the proof that this is an only one in each class.

values of these elements; in one such form the coefficients:

$$a, b, c$$

are always positive, while the coefficient combinations:

$$a'^2 - bc, b'^2 - ac, c'^2 - ab, aa'^2 + bb'^2 + cc'^2 - abc - 2a'b'c' = -D, \quad (2)$$

the last $-D$ of which is called the determinant of the form, are negative. † Owing to these conditions there are always three through the equations:

$$\cos\lambda = \frac{a'}{\sqrt{bc}}, \cos\mu = \frac{b'}{\sqrt{ac}}, \cos\nu = \frac{c'}{\sqrt{ab}}$$

fully determined acute or obtuse angles λ, μ, ν, from which a three-edged corner can be built, here the condition necessary to this:

$$\cos^2\lambda + \cos^2\mu + \cos^2\nu - 2\cos\lambda\cos\mu\cos\nu < 1$$

with $D > 0$ coincided. Here nevertheless with the same angles λ, μ, ν two corners symmetrical to each other could be built, therefore we will agree to always choose the corner from these two, with which the edges, as they lie opposite to these angles in sequence, follow one another from left to right with regard to a straight line directed from the vertex 0 to the inside of the corner which can be thought of as going upward. If we now look at the three edges as the positive axes of a coordinate system we could connect the entire infinite space with our form in which we view the product $x\sqrt{a}, y\sqrt{b}, z\sqrt{c}$ as the coordinates of an arbitrary point of the space, and then φ expresses the square of the distance of this point from the vertex, or more general still the square of the distance of two points, the correspondent coordinates of which have those products to differences.

If one establishes now with three new indeterminate elements x', y', z' the linear expressions:

$$x = \alpha x' + \beta y' + \gamma z', y = \alpha' x' + \beta' y' + \gamma' z', z = \alpha'' x' + \beta'' y' + \gamma'' z', \quad (3)$$

† Disquisitiones arithmeticae, art. 271

of which only one restriction shall take place, that the determinant set up from the 9 coefficients α, β, γ, α', β', γ', α'', β'', γ'':

$$\alpha\beta'\gamma'' + \beta\gamma'\alpha'' + \gamma\alpha'\beta'' - \gamma\beta'\alpha'' - \alpha\gamma'\beta'' - \beta\alpha'\gamma'' = E \qquad (4)$$

is not zero, then φ changes into a new form φ', with regard to which all correspondings shall be indicated with the accented alphabets. If one lets the new form again correspond to an infinite space, then through it two infinite spaces connect point for point with each other, while every two points correspond to each other when in the expressions of their coordinates:

$$x\sqrt{a}, y\sqrt{b}, z\sqrt{c};\ \ x'\sqrt{a'}, y'\sqrt{b'}, z'\sqrt{c'}$$

the elements x, y, z and x', y', z' are linked with one another through the equation (3). If the expressions just written are the coordinate differences for two pairs of corresponding points, then apparently the same relation among $x, y, \ldots$ still holds, out of which from the above and as a result of $\varphi = \varphi'$ it follows immediately that the distance of every two points of a space is equal to the distance of corresponding ones of another. The two spaces connected with each other are therefore either congruent or symmetrical, ie. they can, while the beginning points 0 and $0'$ are laid on each other, come to such a position that either every point falls on its corresponding one or on the opposite point of the latter, when we call for short opposite points two points of the same space which lie from the beginning point at the same distance and in the opposite direction. In order to decide which of these two cases takes place, one has lines to draw in the one space from the vertex to three arbitrary points, and then to investigate whether the straight lines drawn in the other from its vertex to the corresponding points present a corresponding series or the opposite one. If one takes for example in the second space the lines from the points with the coordinates:

$$\sqrt{a'}, 0, 0;\ 0, \sqrt{b'}, 0;\ 0, 0, \sqrt{c'}$$

drawn, lines falling on the positive axes of the second space, then these follow one another from the agreement dealt with above from right to left. For the corresponding points in the first space one has the coordinates:

$$\alpha\sqrt{a'}, \alpha'\sqrt{a'}, \alpha''\sqrt{a'};\ \beta\sqrt{b'}, \beta'\sqrt{b'}, \beta''\sqrt{b'};\ \gamma\sqrt{c'}, \gamma'\sqrt{c'}, \gamma''\sqrt{c'}.$$

In order to determine whether the lines directed to these points follow one another from left to right, ie. as the axes of the first space, or follow in the reverse order, one can make use of the theorem which is known or easily derivable from known properties ‡, from which the straight lines drawn to the three points (ξ, η, ζ), (ξ', η', ζ'), (ξ'', η'', ζ'') present the same series as the axes of ξ, η, ζ or the opposite one, according to the determinant built from the 9 coordinates, when one gives the term $\xi\eta'\zeta''$ in it the positive sign, is positive or negative. For our case this determinant becomes $E\sqrt{a'b'c'}$; therefore congruence symmetry holds according as E is positive or negative.

Til now the elements x, y, z had arbitrary values. If we let them now only further mean integers, then instead of the integral space we have an infinite system of points parallelly arranged, ie. a point system of which through the intersections of three lines parallel equidistant planes would be created. If we assume now further that the substitution coefficients α, β, γ are also integers and E has the values ± 1, then every integral combination x, y, z would represent an integral combination x', x', z' and vice versa. The parallelepipedal systems thus connected with one another would as a result coincides with the other or with the opposite points of the latter. Yet, here the opposite points of points of one such system again make the same system, the two cases are not different from each other, and this becomes evident also from the circumstance that φ' remains unchanged when one takes α, β, γ with opposite signs through which E changes into $-E$. The two systems are therefore always congruent, and one sees that systems which correspond to two equivalent ternary forms φ and φ' are the same spatial structure in two different patterns. Conversely equivalent forms represent any two different parallelepipedal patterns of the same system. If one takes namely any one point of the system as the common starting point, then one has between the coordinates relation to the two axis systems and therefore also between the elements x, y, z, x', y', z' proportional to them linear equations without constant term, ie. equations of the form (3), and here from our supposition, when x, y, z are inte-

‡ Disquisitiones generales circa superficies curvas auctore, C. F. Gauss §2. VII

gers, x', y', z' must also have the same characteristic and vice versa, therefore it follows that $\alpha, \beta, \gamma, \ldots$ are likewise integers and that $E = \pm 1$. On the other hand one has for the homogeneous entire values of the elements the equation $\varphi = \varphi'$, which accordingly also identically takes place, q.e.d.

Similar interrelations occur between a positive binary form:

$$lx^2 + 2mxy + ny^2$$

and a system of points parallelogrammatically arranged. One takes here two axes leant against each other under the angle θ determined through the equation $m = \sqrt{ln}\cos\theta$, while one always invariably proceed with the discrimination of these axes and for example chooses the second on the left-hand side of the first one, after a fixed side of the plane is denoted as the higher one and $x\sqrt{l}, y\sqrt{l}$ viewed as coordinates, one would obtain a system of points completely determined through the quadratic form, which could be considered as the intersection of two series of equidistant parallel lines. If then between two forms the so-called proper equivalence takes place, so that $\alpha\delta - \beta\gamma$ in the substitution equations $x = \alpha x' + \beta y', y = \gamma x' + \delta y'$ is equal to the positive unity, then the corresponding systems can be brought to the coincidence through movement in the plane, while in the other case where $\alpha\delta - \beta\gamma = -1$, to say in general, one of the systems must be shifted for this purpose.

§2

After we have established in the foregoing the connection between the quadratic forms and certain geometrical patterns, there are a few further properties of these patterns to develop, whereby we would for short call a system of points arranged parallelogrammatically or parallelepipedally a *system of second* or *third order*, and infinite series of equidistant points in straight line a *system of first order*.

It seems that the common character of all three types of the system consists in that when such a system is brought into another position through a movement without rotation, which we wish to know a displacement of,

that a point of it changes into the position occupied by another in the beginning, the same happens for all points, therefore that the system in its new position fully coincides with the system in the original one. It can be easily proved that the movability just discussed completely characterises all three types of the systems, and that a system endowed with this characteristic, when it lies in a plane and contains three points not lying in a straight line such that finally a system contains points at least four of which are not found in a plane, will be respectively a system of first, second or third order.

If one has for example a system of points which lie all together in the same straight line, and a and a' are two adjacent points of it, then through a displacement through which a gets to a', a' would get to a'' which is as far from a' as a' is far apart from a; the point a'' therefore also belongs to the system, and the system has no point between a' and a'', here one such point would be known before the movement between a and a'. Here this inspection could be pursued for both sides in the indeterminate, therefore the assertions is proved.

Now let two adjacent points a and a' be in a planar system with the characteristic feature of movability, so that no point of the system is found in the line aa' between a and a'. Here through the displacement from a to a' the infinite straight line aa' moves along by itself, therefore it follows that the entire points of the system in this straight line makes up a system of first order $\ldots{}''a'aaa'a''\ldots$. Here then from the assumption the system still has at least one point outside this straight line, therefore let b be one of the points closest to this straight line. If now enters a displacement through which a gets to b, then the system of first order changes into the new position $\ldots{}''b'bbb'b''\ldots$ and belong in this position to the original system; it is immediately clear that a point of the system can neither be found among the points $\ldots, {}''b, {}'b, b, b', b'', \ldots$ nor among the lines $\ldots{}'bbb'\ldots, \ldots{}'aaa'\ldots$. If one concludes directly, one sees that the entire system can be parallelogrammatically aranged, and that one can choose $aa'b'b$ for a basic parallelogram of it. We further add that through the given construction apparently all parallelogrammat-

ical patterns of which the system is capable could be obtained. It follows from this that the choice of a' up to the obviously necessary restriction that no point lies between a and a' is totally arbitrary, and that b can be taken arbitrarily in the nearest parallel line.

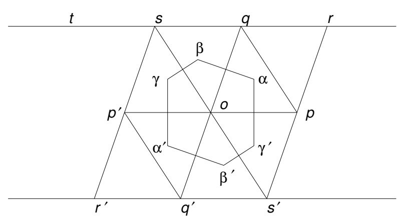
One has finally a system with the characteristic feature of the movability which contains points at least four of which are not lying in the same plane, therefore one lay a plane through any three points of this system not lying in a straight line. Here through any parallel displacement effected with this plane this is moved into itself, consequently points found in this plane build a system of second order from the previous system. As a result one has partitioned this system parallelogrammatically somehow or other, one takes one of the remaining points of the spatial system which lie closest to the plane, and administer a displacement to the system through which an arbitrary point in the plane comes to the point chosen well outside that plane. Through repeated application of this [displacement] and through the movement opposite to it one apparently obtains a parallelepipedal pattern of the given system, and it is immediately clear that the construction specified has the due generality, here the choice of the first plane, the pattern of the system of second order in this plane and finally the choice of points in the neighbouring plane can happen at will.

In the end of this paragraph we will point out further that, as one also partition the same system of second or third order, the parallelogram or parallelepiped lying at the basis of the respective partitioning always retains the same capacity, the geometrical consequence of the sentence is that equivalent forms have the same determinants. If one imagines namely in the plane of a system of second order a line returning to itself, for example a circle line, designates with z the surface area enclosed by it and with s the number of points in the inside of the line, in the course of which it makes no difference whether one wants to include the points on the periphery or not, then obviously the quotient $\frac{z}{s}$ has with growing radius the capacity of a basic parallelogram to the boundary, from which, here s and z are independent

of the type of the pattern, the theorem for systems of second order becomes evident. Totally in the same fashion follows the soundness of the assertions for spatial systems.

§3

We will now point out that a system of second order would always admit partitioning by a basic parallelogram, the sides of which are not larger than its diagonals.



I. Let o be an arbitrary point of the system. The remaining points of this system always lie pairwise in the same distance and opposite direction from o. Now let p be one of the points of the pairs, for which the distance from o is smaller than for every other pair. The same smallest distance holds for more than one pair, therefore one would choose p at will in one of these. The given system consists of an infinite quantity of systems of first order congruent among one another and of the same distance, one of which is that for which o and p belong. In one of the two adjacent to this latter one, one takes the point q which is next to o, or, supposing the same shortest distance should occur for two points, arbitrarily takes one of the two. The parallelogram $poqr$ thus obtained has the desired property, here in accordance with the construction $op \leqq oq$, $oq \leqq or$, $oq \leqq os = pq$. A basic parallelogram which satisfies these conditions shall be called a reduced one.

II. We have now the relation between one such parallelogram and the planar system in which it belongs to establish. If $poqr$ is a reduced parallelogram we would be able to, without breaking the generality, assume the

angle poq as not obtuse, here in the opposite case the angle at o for the parallelogram adjacent to the same structure is an acute one, and likewise we could assume $op \leqq oq$. Thereupon $or > oq$ is apparent, and we have only the condition $pq \geqq oq$ still to consider. If this is supposed and if we put for the reduction $op = \sqrt{l}$, $oq = \sqrt{n}$, so that consequently $l \leqq n$ the connection of our parallelogram to the entire point system would possibly be described to the effect that the minimum of the distance of any point of the system from o is equal to $\sqrt{l}$, and that after one has chosen a point at this distance, in all distances still remaining, ie. outside the straight line drawn from o to the former one, the second minimum is equal to $\sqrt{n}$. The precisely stated holds true all in general; what we now add, that namely the first minimum only occurs for the point p (when we always only choose one of two opposite points), the second only for q, holds true with the following exceptions:

1. If $op < oq$, $oq = pq = os$, then the first minimum takes place only for p, the second for q and s.

2. If $op = oq$, $oq < pq = os$, then the minima are equal, and one can exchange p and q with one another.

3. If finally $op = oq = pq = os$, then one can choose one of the points p, q, s as first point and then one of the remaining as second one.

In order to demonstrate the precisely asserted, we have obviously, the opposite points are always equally far from o, only to point out that q lies closer to o, *firstly* than all remaining points in the straight line sqr, with exception of the point s, the distance from o of which according to the assumption is equal to $os = pq \geqq oq$, and *secondly* than all points of the subsequent parallel lines.

Here $pq \geqq op$, $pq \geqq oq$ and the angle poq is not obtuse, therefore the triangle opq and consequently also the oqs congruent with it has no obtuse angle; therefore the perpendicular dropped from o on qs lies between s and q (inclusive), with which

one sets:

$$\cos poq = \frac{m}{\sqrt{ln}},$$

where consequently m is not negative, therefore one has:

$$\overline{pq}^2 = l - 2m + n \geqq \overline{oq}^2 = n,$$

and hence:

$$2m \leqq l, 2m \leqq n, 4m^2 \leqq ln.$$

If one further sets the square of the height of our parallelogram ($op = \sqrt{l}$ regarded as base line) equals k, one obtains for the square Δ of its volume:

$$\Delta = lk = ln - m^2 \geqq \frac{3}{4} ln,$$

and hence:

$$\sqrt{k} \geqq \frac{1}{2}\sqrt{3n}.$$

According to this the second line is at least $\sqrt{3n} = oq\sqrt{3}$ away, and the second point is also established.

III. Here the successive minima $\sqrt{l}, \sqrt{n}$ are decided through the system as such and are independent of any fixed pattern and on the other hand, as we have just seen, correspond in quantity with the sides of the reduced parallelogram, therefore one sees that when the system permits various patterns of this fashion, the sides of the reduced parallelograms will always contain the value $\sqrt{l}$ and $\sqrt{n}$. One would essentially obtain as a result all possible basic parallelograms if one draws lines from o to all adjacent points (always with exemption of the opposite points) and then takes the nearest or the two nearest points in one of the respective nearest parallel lines; and here from the definitively demonstrated (II) this nearest or these nearest points lie closer to o than all points of the subsequent parallel lines, therefore one can see from the condition that the second points are to be taken in the first parallel line. Therefore all possible patterns would be produced if one successively connect o with all point pairs for which the successive minima take place, from which at once follows with consideration from (II) that in general and in the second one of the singular cases obtained then there is only one such pattern, in the first and third exception case however

there are respectively two and three patterns of the systems.

In our present reference the precisely obtained singular cases correspond with the suppositions $2m = l < n$, $2m < l = n$, $2m = l = n$.

§4

We have so far only dealt with properties of the geometrical structures which is to be looked at from the theory of forms as the constructive representation of well known theorems and are already indicated in the article cited in the introduction. It is now to solve another problem of another kind, the problem namely when a system of second order is given and a fixed point o of it is examined, to determine the part of the plane every point of which lies nearer to o than to any other points of the system. Here the condition that a point does not lie farther from o than from any other v, therein consists that the point with o on the same side of the perpendicular drawn up in the middle of ov, so we would consequently have o to combine with all remaining points of the system and the convex polygon built from all corresponding perpendiculars to construct. But from these perpendiculars in infinite quantity only a limited number comes into question, while the remaining ones do not meet the polygon determined by it. We abide by all suppositions attended to, so that consequently $op \leqq oq$ in the reduced parallelogram $(poqr)$, the angle poq is not obtuse and opq, oqp are acute. This supposes, it is easy to understand, that one has only the six vertices p, q, s, p', q', s' of the four parallelograms meeting at o to take into consideration, and that the perpendiculars corresponding to s and s' and the building diagram in the particular case, when poq is a right angle, only touch, which then the same happens for the perpendiculars corresponding to r and r'. If one draw the straight line pq, os, $p'q'$, os', one obtains the congruent triangles:

$$poq, qos, sop', p'oq', q'os', s'op.$$

If one consider only the points p, q, s, p', q', s', one has to draw a perpendicular in the middle of the straight lines going from o to these points, ie the same construction to

make as when one wished t find the middle point of circumscribed circles for the designated triangles. Here no obtuse angle is found in the triangles, therefore each two successive perpendiculars not outside the corresponding triangle intersect. One obtains therefore the hexagon $\alpha\beta\gamma\alpha'\beta', \gamma'$ with the centre o and equal opposite angles and sides as the space, inside of which every point is less far apart from o than from one of the points p, q, s, p', q', s', and one is easily convinced that, with exception of r and r', the perpendiculars corresponding to the remaining points do not meet our hexagon. This requires, as a result of symmetry, only for the points in and above the line pop' to be established. For the former ones it is clear; for the latter ones it would hence appear that their distance from o is larger than the diameter of the circle traced around the hexagon. If one designate the square of its radius ρ, then:

$$4\rho\Delta = ln(l - 2m + n),$$

from which as a result of $2m \leqq l, 2m \leqq n, \Delta \geqq \frac{3}{4}ln$, it follows:

$$4\rho \leqq \frac{4}{3}(l - 2m + n) \leqq \frac{8}{3}n.$$

Here now for the points of the second and the subsequent parallel lines, as already remarked, the square of their distance from o amounts to at least $3n$, therefore there still remain simply the points in $tsqr$ apart from s, q, r to examine. From all of these none is closer to o than t, for which the square of the distance is equal to $4l - 4m + n$, and that this is larger than 4ρ, one immediately sees when one multiplies with Δ and then looks at the inequalities $2m \leqq l \leqq n$. As for the point r, one is also convinced by the same manner that the square of its distance from o is equal to $l + 2m + n > 4\rho$, the only case excluded, where $m = 0$, in which the corresponding perpendicular touches. It is thus demonstrated that every point in the inside of the hexagon $\alpha\beta\gamma\alpha'\beta'\gamma'$, and only one such hexagon, lies closer to the point o than any other of the system. On any side the distance from o would be equal to the distance from a second point, which for example for $\alpha\beta$ is the point q, and every vertex of the diagram is of the same distance from o and another two other points of the the system. The latter statement undergoes a modification only in the special

cases when the angle poq is a right angle; thereupon β and γ as well as β' and γ' coincide, and the hexagon turns into a rectangle, of which the corner from o and another three other points of the system are equally far apart.

It goes without saying that one will always obtain the same hexagon whose reduced parallelogram one also lay as foundation of the construction in the singular cases, where more than one exists, just as also that the hexagon or quadrangle corresponding to all the points of the system are congruent and cover the whole plane of it.

We notice further that, as one is easily convinced, the expression:

$$\rho = \frac{ln(l-2m+n)}{4(ln-m^2)}$$

decreases when one therein, assuming l and n constant, allows m to grow from zero up to its limit $\frac{1}{2}l$, so that consequently:

$$\rho \leqq \frac{1}{4}(l+n) \leqq \frac{1}{2}n. \tag{1}$$

Also in addition the following inequality takes place:

$$2\Delta(n-\rho) \geqq ln^2, \tag{2}$$

the soundness of which is immediately evident when one multiplies with 2, moves everything to one side and then applies $\Delta = ln - m^2$, $4\Delta\rho = ln(l-2m+n)$, by the mean of which it changes into $ln(l-2m) + 2mn(n-l) \geqq 0$.

§5

We come now to our true topic and have to prove that every system of third order can be arranged according to a parallelepiped whose faces are reduced parallelograms and whose edges, from which every four are equal to one another, do not exceed their diagonals.

After one has fixed an arbitrary point (0) of the systems, one would choose in pairs of opposite points for which the distance from (0) is a minimum, or when the minimum of the distance exists for several pairs, would

arbitrarily choose a point (1) in one of these pairs. From all points outside the straight line (01) one would again choose one of the two nearest (2), through which again the selection under several pairs, for which the same shortest distance takes place, can be arbitrarily made. Here in the whole system, with exception of the points in (01), no point lies closer to (0) than to (2), so the same is valid also for the plane (102), and (102) is a reduced parallelogram for the system which contains this plane (§3, III). One now takes in one of the two nearest parallel planes the point which is closest to (0) or, when the minimum occurs for more than one, one of the nearest ones and connect (0) with the chosen point (3), therefore the parallelepiped would with the edges (01), (02), (03), as is easy to see, suffice the requirement. Next from the construction it follows: $(01) \leqq (02) \leqq (03)$. Here for the bases of the parallelepiped (we would always indicate as such each face opposite to one another in which are found edges two of which do not exceed the third one in size, and the term side faces apply to the four remaining ones) it is already proven that they are reduced, therefore we have in virtue of the precisely noted doubled inequality only to point out further that the four diagonals of the side faces, just as the four diagonals of the body, are not smaller than (03). Now the eight diagonals mentioned above will agree, as one immediately sees, in size with the eight connecting lines which could be drawn from (0) to the eight points lying around (3) in the plane of the higher bases if we indicate this way for convenience the eight vertices of the four parallelograms meeting at (3). The fact that from the afore-mentioned connection lines none is smaller than (03) follows from the condition by which (3) is being chosen.

After we have convinced ourselves that a system of third order can always be partitioned by a reduced parallelepiped, we now have to establish the relation between such parallelepiped and the system and particularly to compare the distance of the point of systems from (0) with one another. We set $(01) = \sqrt{a}, (02) = \sqrt{b}, (03) = \sqrt{c}$ and always hold fast the assumption $a \leqq b \leqq c$.

1. In the plane of the base the conditions discussed above (§3, II) occurs, so that consequently the successive

minima of the distance are always $\sqrt{a}, \sqrt{b}$ in size, whereby then in the singular cases mentioned there there is an arbitrariness in the choice of the points.

2. We now look at the points outside the plane of the base underneath namely first of all the one in the plane of the base above. Here from the assumption that our parallelepiped is a reduced one, the line (03) is not longer than one of the straight lines drawn from (0) to the eight points lying around (3), so as a result the foot of the perpendicular dropped from (0) on to the plane of the base above would not be farther apart from (3) than from one of the eight points mentioned. This foot point therefore does not fall outside the hexagon or quadrangle constructed to belong to (3) in the last paragraph. Of those eight points can exceptionally, when the foot point falls on one side, one, or it could, when the foot point meets with a vertex, two (three, when the polygon becomes a rectangle,) lie equally close to the foot point as the point (3), while all remaining points of the plane are further apart from that foot point. It follows from this that the shortest distance (amounting to $\sqrt{c}$) from (0) to the a point in the base above is valid in general only for the point (3), but can exceptionally take place for one, two or even three other points.

3. For the consideration of the following parallel planes we have a boundary for the square h of the perpendicular already mentioned to determine. Here the foot point of the perpendicular does not fall outside the hexagon which belongs to (3), therefore, when ρ denotes the square of the radius of the circumscribed circle:

$$h \geqq c - \rho.$$

Now also from §4: $\rho \leqq \frac{1}{2}b \leqq \frac{1}{2}c$, consequently $h \geqq \frac{1}{2}c$. Here therefore the second parallel plane is at least $\sqrt{2c}$ away, therefore there is over the higher base only points, the distance from (0) of which is greater than $\sqrt{c}$.

If one summarises what has been said, one will see that the minimum of the distance for the entire system has the value $\sqrt{a}$, that, after a point is chosen at this distance, the minimum in the still remaining directions

amounts to $\sqrt{b}$, and that finally after the second point is also fixed, for all points outside the plane, which is determined through (0) and the first two points, the smallest distance from (0) is reduced to $\sqrt{c}$. If the successive minima $\sqrt{a}, \sqrt{b}, \sqrt{c}$ are also always completely determined in quatity, the same minimum in local relation will not be true without several exceptions which are easy to specify. If for example $a \leqq b$, $b < c$, the first two points are to be chosen in the lower base, whereby the singular cases mentioned in §3, II could occur, while the third point lies in the higher base, has a fixed position there in general, in singular cases however can occupy two, three or four different places. One ever so easily overlook that varieties in the other two cases, where $a < b = c$ or $a = b = c$, could happen.

Here from the assumption of a reduced parallelepiped with the edges $\sqrt{a} \leqq \sqrt{b} \leqq \sqrt{c}$ the length of these edges have yielded themselves as the successive minima of the system, thus it immediately follows that when several reduced parallelepiped exist from which the system can be arranged, these all become in agreement with one another with regard to the lengths of their edges, and it could also be easily pointed out that three of the lines directed from (0) to points of the systems of the lengths $\sqrt{a}, \sqrt{b}, \sqrt{c}$ when they only do not lie in the plane, are always the edges of a reduced parallelepiped. It requires therefore only the easy consideration already applied in a similar case (§3, III). Here after this the entire reduced parallelepipeds would be obtained when one construct the successive minima of every possible types, therefore it becomes evident that when this can happen in only one way (to which we also consider the case where, with the equation of two of the quantities $\sqrt{a}, \sqrt{b}, \sqrt{c}$ or with the equation of all three, the three lines are locally completely determined and only an exchange between two or all three can occur) that spatial system would allow only one pattern from a reduced parallelepiped. In all other cases there are several such patterns, the parallelepipeds of which form the basis, which could be either all different from one another or only different in part or even could be all congruent to one another. (Similarly in the two singular cases of a system of second order mentioned above the reduced parallelograms underlying the two or three various patterns were congruent to one another.)

To the determination of the question whether a system of third order permits only one or more than one pattern from a reduced parallelepiped, it would consequently only need the knowledge of a single pattern of the system, and the first case would always and exclusively take place when the reduced parallelepiped given through this pattern is of such a property that all lines which can not be exceeded by others actually exceed this parallelepiped, that is when all diagonals of the faces are larger than their sides and all diagonals of the parallelepiped are similarly larger than the edges of the bodies.

§6

As we now apply the results of the last paragraph to the ternary form, shall the uniformity be assumed because of and in order to avoid pointless differentiation, that one has given every ternary form:

$$ax^2 + by^2 + cz^2 + 2a'yz + 2b'xz + 2c'xy \qquad (1)$$

through transposition or change of sign of indeterminate elements, as a result of which the form does not belong to the same class, a single form, that firstly $a \leq b \leq c$ holds, that secondly under the coefficients a', b', c', when not the case that all of them are nonzero and are negative, none has the negative sign, and thirdly, when $b = c$ holds, c' apart from the sign not greater than b', when $a = b$ holds, b' not greater than a', and lastly when $a = b = c$ holds, neither c' greater than b' nor b' greater than a' holds. As is easy to see these condition can only be satisfied in one way and their introduction gives the advantage that, as already without these conditions every ternary form corresponds with a completely determined parallelepiped, now to every parallelepiped also belongs an analytical expression the coefficients of which are also completely determined with regard to their sequence and their signs. This assumed, we mention the form (1) in which $a \leq b \leq c$ also holds, a reduced one, when it corresponds with a reduced parallelepiped. There the diagonal of the area must not be smaller than the sides themselves, so one has:

$$a \pm 2c' + b \geq b, \; a \pm 2b' + c \geq c, \; b \pm 2a' + c \geq c.$$

One sets $\sigma = -1$, where a', b', c' are all three negative, otherwise $\sigma = 1$, so these conditions are synonymous with:

$$a \geq 2c'\sigma,\ a \geq 2b'\sigma,\ b \geq 2a'\sigma, \tag{2}$$

and only, when the equal sign holds, would one of the diagonals in the corresponding parallelogram be equal to a side. The conditions with regard to the diagonals of the parallelepiped result in:

$$a + b + c + 2a'\epsilon + 2b'\delta + 2c'\delta\epsilon \geq c \quad (\delta = \pm 1, \epsilon = \pm 1),$$

where the signs in $\delta = \pm$, $\epsilon = \pm 1$ are arbitrary. One look next at the case where none of the coefficients a', b', c' is negative, and take into account the four sign combinations, as well as that, when a and b are equal to one another then $b' < a'$, so one immediately sees that our inequality is by itself capable of always meet the condition contained, and that the limiting case of the equation in which the diagonals of the edge $\sqrt{c}$ become equal, only once and only then can it occur, when one of the quantities b', c' is equals to zero and when at the same time of the conditions (2) the two quantities b', c' of which are relating to one another, as well as the one which contains a', satisfies the limiting case of the equation. a', b', c' are negative, then our inequality is always fulfilled that the limiting case can not take place, except when $\delta = \epsilon = 1$, so that the consequently the new condition is established:

$$a + b + 2a' + 2b' + 2c' \geq 0, \tag{3}$$

where again the lower sign relates to the equality between a diagonal and the edge $\sqrt{c}$.

The condition just developed (2) and, when a', b', c' are negative, (3) above is are therefore fulfilled, that the inequality takes place in none of the inequalities of the limiting case, therefore in the class to which the form belongs it would not give a second one of these various ones with or without equality signs in the definition condition, here according to at the end of the last paragraph notice that the corresponding system of points can only be partitioned from a reduced parallelepiped. The matter stands differently when the upper signs do not

take place in all conditions; there could then occur in the same class several reduced forms that can be derived from a given one. It is sufficient to demonstrate this for a main case. We choose for this the case where $b < c$.

Next one assumes $a > 2c'\sigma$, therefore the direction of the edge $\sqrt{c}$ can only be altered when there are namely in the plane of higher base still one or more points, the distance from the vertex of which amounts to $\sqrt{c}$. When ξ, η are 1 the one such points corresponding values of the element so would, when the third edge depends on it, all the coefficients except a', b' remain unchanged, these respectively change into $a' + c'\xi + b\eta$, $b' + a\xi + c'\eta$ as one is easily and almost convinced without calculation. Now from the specification made earlier are the values of ξ, η which meet the condition, when $a = 2b'\sigma$:

$$\xi = -\sigma, \ \ \eta = 0;$$

when $b = 2a'\sigma$:

$$\xi = 0, \ \ \eta = -\sigma;$$

when simultaneously $a = 2b', b = 2a', c' = 0$:

$$\xi = -1, \ \ \eta = -1;$$

and when a', b', c' are negative and the equation $a + b + 2a' + 2b' + 2c' = 0$ complied with:

$$\xi = 1, \ \ \eta = 1.$$

Corresponding to these four assumptions one has consequently transformed a', b' into:

$$a' - c'\sigma, \ b' - a\sigma (= -b');$$

$$-a', \ \ b' - c'\sigma;$$

$$-a', \ \ -b';$$

$$a' + b + c', \ a + b' + c'.$$

From the third case and generally from the assumption $c' = 0$ one can foresee, here this represents a new form which afterwards one has undertaken in the same one the change of sign stated in the beginning of this paragraph, apparently with the form from which one has derived,

become identical. In each of the three remaining assumptions one obtains from application of the necessary sign change a new reduced form belonging to the same class (provided that it does not coincide with the original one), and one obtains two such forms when two of our assumptions hold at the same time. With this then the specification of the form is brought to an end, here apparently the simultaneity of all three assumptions can not take place. If, always under the assumption $b < c$, $a = 2c'\sigma$, one would have, provided that $a < b$, the two edges rotated in the base, for $a = b$ the first edge can also pass into the original position of the two first edges and this new position or both of these new positions of the first two edges must be associated with all the directions of the third one, the original one not excluded.

One can thereby easily remove the inconvenience that in singular cases several reduced forms can be associated in the same class, and thereby take away the exception that for such singular cases in general definition one still include the known secondary conditions which can be proved when one for instance set up the demand that the last coefficient c', provided that it is not fully fixed, maintains the smallest numerical value of which it is capable in the reduced forms of the class, and then likewise with regard to b'. For this to notify an example, we will examine it under the singular cases dealt with earlier where $b < c$, from the three conditions (2) none with the lower signs holds, however the three negative values a', b', c' satisfy the equation:

$$a + b + 2a' + 2b' + 2c' = 0.$$

From the previous observations c' is fixed, and there exists for this case only two reduced forms. a' and b' are the values of the forth and the fifth coefficients in one of them, therefore they are in the other $a' + b + c'$, $a + b' + c'$, or, here the last value are apparently positive, c' is negative and consequently, in order for the sign specification to be sufficient, z to transform into $-z$, rather $-(a' + b + c')$, $-(a + b' + c')$. As it is quite natural, these values suffice when one substitute them for a', b', again the equation:

$$a + b + 2a' + 2b' + 2c' = 0,$$

and from them come the values a', b' in the same manner, as they themselves are originated from a', b'. Here according to this the fifth coefficient admits only the two negative values b' and $-(a + b' + c')$, their sum equals $-a - c'$, therefore one see that when one further add to the definition conditions:

$$-b' \leqq \frac{1}{2}(a + c'),$$

the class would contain only one reduced form.

While we conclude the essay, we will still from our principles derive a beautiful theorem, found by *Seeber* through induction and demonstrated by *Gauss* in the announcement already often quoted. From this theorem in a reduced form the production of the first three coefficients is not larger than the doubled absolute value of the determinant.

Here the absolute value of the determinant is equal to the square of the volume of the parallelepiped corresponding to the form, thus consequently, from the expression employed in §5, 3 the inequality to be proved:

$$abc \leqq 2\Delta h,$$

where Δ represents the square of the base. One sets:

$$c = b + t,$$

where consequently t is not negative, draws off from the inequality obtained in §5, 3:

$$h \geqq c - \rho = b - \rho + t,$$

from which one has multiplied it with 2Δ, the equation:

$$abc = ab^2 + abt,$$

thus one obtains:

$$2\Delta h - abc \geqq 2\Delta(b - \rho) - ab^2 + (2\Delta - ab)t.$$

Here now from the inequality at the end of §4, $2\Delta(b - \rho) - ab^2$ is not negative and $2\Delta - ab \geqq \frac{1}{2}ab$ is positive, therefore the truth of the theorem becomes evident.